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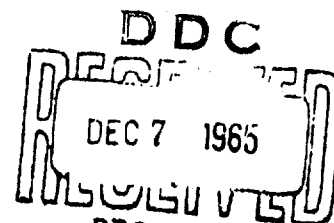


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Numerical Solution of Flood Prediction and River Regulation Problems



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and A. TROESCH

REPORT II

NUMERICAL SOLUTION OF FLOOD PROBLEMS IN
SIMPLIFIED MODELS OF THE OHIO RIVER AND THE
JUNCTION OF THE OHIO AND MISSISSIPPI RIVERS.
CONCLUSIONS VALUABLE FOR THE ACTUAL CASES.

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OHIO RIVER DIVISION

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NEW YORK UNIVERSITY
Institute of Mathematical Sciences

NUMERICAL SOLUTION OF FLOOD PREDICTION AND
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E. Isaacson, J. J. Stoker, and A. Troesch

Report II

Numerical Solution of Flood Problems in Simplified
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Ohio and Mississippi Rivers. Conclusions Valuable
for the Actual Cases

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NUMERICAL SOLUTION OF FLOOD PREDICTION AND
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E. Isaacson, J. J. Stoker, A. Troesch

II. Numerical Solution of Flood Problems in Simplified Models of the Ohio River and the Junction of the Ohio and Mississippi Rivers. Conclusions Valuable for the Actual Cases.

1. Purpose of the present report and summary of its results.

In Report I a mathematical basis was laid for the numerical solution of flow problems in rivers. The ultimate aim is to carry out numerically the solution of a flood problem for the Ohio River in a concrete case by using an appropriate digital computer, to compare the results with the observations, and, in general, to study the feasibility of such methods of attacking this type of problem; in addition, the problem of floods at the junction of the Ohio and the Mississippi, and problems concerning the regulation of the Tennessee River through controls at the Kentucky Dam are to be solved numerically. In all of these cases it is necessary to make use of a considerable bulk of observational data--cross-sections and slopes of the channels, measurements of river depths and discharges as functions of time and distance down the river, drainage areas, observed flows from tributaries, etc.--in order to obtain the information necessary to fix the coefficients of the differential equations derived in Report I and to fix the initial and boundary conditions. This in itself is a task with some complexities, and it also takes time and

the cooperation of several groups of people. In the meantime, therefore, it was thought wise to try out the numerical methods proposed in Report I on three problems which are simplified versions of actual problems, and to carry out the solutions by using ordinary hand calculators. The present Report II has as its purpose the presentation of the solutions in these special cases, together with an analysis of their bearing on the concrete problems for actual rivers.

The models chosen correspond in a rough general way (a) to two types of flow for the Ohio and (b) to the Ohio and Mississippi at their junction. Rivers of constant slope, with rectangular cross-sections having a uniform breadth and roughness coefficient are assumed. In this way differential equations with constant coefficients result. The values of these quantities are, however, taken to correspond in order of magnitude with those for the actual rivers. In the model of the Ohio, for example, the slope of the channel was assumed to be 0.5 ft./mile, the quantity n (the roughness coefficient in Manning's formula) was given the value 0.03, and the breadth of the river was taken as 1000 feet. (These values were recommended to us by the engineers on the basis of their knowledge of the Ohio.) It was assumed that a steady uniform flow with a depth of 20 ft. existed at the initial instant $t = 0$, and that for $t > 0$ the depth of the water was increased at the point $x = 0$ from 20 ft. to 40 ft. within 4 hours and then held fixed at the latter value. The problem was to determine the flow downstream, i.e. the depth y and flow velocity v as functions of x (for $x > 0$) and t . The resulting

stage and discharge profiles are given for several times in Figures 1 and 2. In a similar fashion the model for the junction of the Ohio and Mississippi was set up. (Figures 4 and 5 display the stage and discharge profiles for various times.)

Much valuable information and insight was gained from the study of these models, especially from the model of the Ohio. This will be discussed in detail later on, but a number of observations might be made at this point. In the first place, the fact that the solution of the simplified flood problem in the Ohio could be carried out numerically by hand computation over a considerable range of values of the distance and time (values at 900 net points in the x, t -plane were determined by finite differences) shows that the problems are well within the capacity of modern calculating equipment. It might be added that the special case chosen for a flood in the Ohio was one in which the rate of rise at the starting point upstream was extremely high (5 feet per hour, in comparison with the rate of rise during the big flood of 1945 which was never larger than 0.7 feet per hour at Wheeling, West Virginia), so that a rather severe test of the finite difference method was made, in view of the rapid changes of the basic quantities in space and time.

The decisive point in estimating the magnitude of the computational work in using finite differences is the number of net points needed, and our model indicates that an interval Δx not smaller than 10 miles along the river and an interval

Δt not less than 0.3 hours in time in a rectangular net in the x, t -plane will yield results that are sufficiently accurate. (Of course, the actual problems for the Ohio will involve empirical coefficients in the differential equations and other empirical data, which will have to be coded for calculating machines, but this will have no great effect on these estimates for Δx and might under extreme flood conditions reduce Δt by a factor of .5.) In our model calculations we used a uniform space interval Δx , along the river's length, but we are investigating the feasibility of using the location of the gaging stations as net points. In our model calculations, we experiment with various finite difference methods, interval sizes and analytic approximations in order to get the practical experience necessary to select the technique to be used for automatic machine computation. We plan to use our ideas in some trial desk calculations made with the physical data for the Ohio and the 1945 flood (the basic data have been provided by the Corps of Engineers and is now being reduced at New York University to a form suitable for computation).

A second computation* for the Ohio model was performed for a steady progressing wave (40 ft. stage upstream and 20 ft. stage downstream). This work indicated that an interval Δx of 10 miles yielded a stage prediction accurate to within 0.6 per cent in a 7 hour forecast.

* To provide with a check on the accuracy of our computational methods, in a problem which could be solved analytically (see sec. 2.4 and App. II).

As a model for the problem of calculating what happens at the junction of two major streams, we worked with three separate river stretches (e.g. the Ohio, the upstream side of the Mississippi, the downstream side of the Mississippi). For each region the same finite difference scheme was used as in the model of the Ohio. In addition, it was necessary to determine the values of stage and discharge at the common junction point from the knowledge that the levels of the three branches agree there and that the water which flows into the junction also flows out.

A much better understanding was gained from the calculations for the model of the Ohio of the relation between the methods used by the engineers in the Ohio River Division in Cincinnati (and other engineers as well) for predicting flood stages, and the methods explained in Report I, which make use of the basic differential equations. At first sight the two methods seem to have very little in common, though both, in the last analysis, must be based on the laws of conservation of mass and momentum; indeed, in one important respect they even seem to be somewhat contradictory. The methods used in engineering practice (which make no direct use of any differential equations) tacitly assume that a flood wave in a long river such as the Ohio propagates only in the downstream direction, while the basic theory of the differential equations we use (as was explained in Report I) tells us that a disturbance at any point in a river flowing at sub-critical speed (the normal case in general and always the case for the

Ohio) will propagate as a wave traveling upstream as well as downstream. Not only that, the speed of propagation relative to the flowing stream, as defined by the differential equations, is \sqrt{gy} for small disturbances and this is very much larger than the propagation speed obtained by the engineers for their flood wave traveling downstream. There is, however, no real discrepancy. The method used by the engineers can be interpreted as a method which yields solutions of the differential equations, with certain terms neglected, that are good approximations (though not under all circumstances, it seems) to the actual solutions in some cases, among them that of flood waves in a river such as the Ohio. The neglect of terms in the differential equations in this approximate theory is so drastic as to make the theory of characteristics, from which the properties of the solutions of the differential equations were derived in Report I, no longer available. The numerical solution presented here of the differential equations for a flood wave in a model of the Ohio yields, as we have said, a wave the front of which travels downstream at the speed \sqrt{gy} , but the amplitude of this forerunner is quite small, while the portion of the wave with an amplitude in the range of practical interest is found by this method to travel with essentially the same speed as would be determined by the engineer's approximate method. What seems to happen is the following: small forerunners of a disturbance travel with the

speed \sqrt{gy} ^{*} relative to the flowing stream, but the friction forces act in such a way as to decrease the speed of the main portion of the disturbance far below the values given by \sqrt{gy} , i.e. to a value corresponding roughly to the speed of a steady progressing wave ^{**} that travels unchanged in form. (One could also interpret the engineering method as one based on the assumption that the waves encountered in practice differ but little from steady progressing waves.)

This analysis of the relation between the methods proposed in Report I and those commonly used in engineering practice indicates why it may be that the latter methods, while they furnish good results in the most important cases, fail to mirror the observed occurrences in other cases. For example, the problem of what happens at a major junction, and various problems arising in connection with the operation of such a dam as the Kentucky Dam in the Tennessee River seem to be cases in which the engineering methods do not work well. These are, in all likelihood, cases in which the motions of interest depart too much from those of steady progressing waves, and cases in which the propagation of waves upstream is as vital as the propagation downstream. Thus at a major junction it is

^{*} As can be seen from Report I, the propagation speed, \sqrt{gy} , which is determined without reference to the friction term, would not be affected by any change in the order of magnitude of the friction force.

^{**} A steady progressing wave is a flow in which the depth and discharge are functions only of $(x - Ut)$, with U a constant velocity. The propagation speed U is determined from the friction term (see sec. 2.4).

clear that considerable effects on the upstream side of a main stream are to be expected when a large flow from a tributary occurs. In the same way, a dam in a stream (or any obstruction, or change in cross-section, etc.) causes reflection of waves upstream, and neglect of such reflections might well cause serious errors on some occasions. On the other hand, the method of numerical solution based directly on the differential equations is applicable in any of these cases, and it seems reasonable, as we propose doing, to test it out as a practical method.

The above general description of what happens when a flood wave starts down a long stream--in particular, that it has a lengthy front portion which travels fast, but has a small amplitude, while the main part of the disturbance moves much more slowly--has an important bearing on the question of the proper approach to the numerical solution by finite differences. It is, as we shall see shortly, necessary to calculate--or else estimate in some way--the motion up to the front of the disturbance in order to be in a position to calculate it at the places and times where the disturbances are large enough to be of practical interest. This means that a large number of net points in the finite difference mesh in the x,t -plane lie in regions where the solution is not of great interest. Since the fixing of the solution in these regions costs as much effort as for the regions of greater interest, the differential equation method is at a certain disadvantage by comparison with the conventional method in such a case. However, it is possible to

determine analytically the character of the front of the wave and thus estimate accurately the places and times at which the wave amplitude is so small as to be negligible; these regions can then be regarded as belonging to the regions of the x, t -plane where the flow is undisturbed, with a corresponding reduction in the number of net points at which the solutions must be calculated. A method which can be used for this purpose has been devised by G. Whitham and A. Troesch, and a description of it is given in Appendix I of this report. If a modern high speed digital computer were to be used, however, it would not matter very much whether the extra net points in the front portion of the wave were to be included or not: many such machines have ample capacity to carry out the necessary calculations.

In sec. 2 a description of the calculations made for the model of the Ohio is given, including a discussion of various difficulties which occurred for the flood wave problem near the front of the disturbance, and particularly at the beginning of the wave motion (i.e. near $x = 0$, $t = 0$), and an enumeration of the features of the calculation which are certain to play a similar role in the more complicated case of the Ohio as it really is. In sec. 3 we give a description of the method used and the calculations made for a model of a flood coming down the Ohio and its effect on passing into the Mississippi. In Appendix I, as was noted above, the method of G. Whitham and A. Troesch for dealing with the front portion of a wave traveling down a river is described. In Appendix II, a discussion of

steady flows and of steady progressing waves is given. These flows were used to determine convenient initial conditions, to test our computing procedures and to furnish the asymptotic behavior (for large time) of the flows.

Furthermore, in order to understand more fully the general behavior of flood waves in open channels and especially why the speed \sqrt{gy} is not the observed propagation velocity for normal flood problems, some theoretical investigations* using the concepts of steady and steady progressing flow were carried out by G. Morikawa, A. Troesch, and G. Whitham and will be reported on later. Their conclusions also have important bearing on the practicability of making flood forecasts for extended periods of time, from the given data.

2. Model of the Ohio.

2.1 Formulation of the flood problem. We consider a simple model which approximates the average characteristics of the Ohio River. The river was assumed to have a rectangular cross section 1000 ft. in width. The slope S of the river bottom was taken as .5 ft./mile and Manning's formula was used for the friction slope S_f ,

* These are related to, and extend, the work of Hayashi, T., Mathematical theory of flood waves. Proc. of the 1st. Japan Nat'l. Cong. for Appl. Mech., 1951.
Deymié, Ph., Propagation d'une intumescence allongée (problème aval). Int. Cong. for Appl. Mech., 5th, Proceedings, Sept. 1938, N. Y., Wiley, 1939.

$$S_f = \frac{v|v|}{\gamma \left(\frac{y}{1 + \frac{2y}{B}} \right)^{4/3}}$$

(since the hydraulic radius R for a rectangular channel of width B and water of depth y is

$$R = \frac{yB}{B + 2y} = \frac{y}{1 + \frac{2y}{B}})$$

The choice

$$\gamma = \left(\frac{1.49}{n} \right)^2 = 2500 \frac{(\text{ft})^{2/3}}{(\text{sec})^2},$$

where Manning's $n \approx .03$, was suggested by the engineers. The differential equations are taken in the form (see p. 14, Report I, eq. (3.6))

$$2cc_x + v_t + vv_x + E = 0 \quad (2.1)$$

$$2c_t + 2vc_x + cv_x = 0$$

where

$$E = -gS + gS_f, \quad v = \text{velocity}, \quad y = \text{depth of water},$$

$$c = \sqrt{gy} \quad (\text{the propagation speed of small disturbances}).$$

In carrying out the computation it was found convenient to use the mile as unit of length and the hour as unit of time.

The physical problem considered was the following: at time $t = 0$, steady flow of depth 20 ft. is assumed. At the "head", $x = 0$, of the river, we impose a linear increase of depth which brings the level to 40 ft. in 4 hours. We then

maintain the level of 40 ft. at $x = 0$. The initial velocity of the water corresponding to a uniform flow of depth $y_0 = 20$ ft. is calculated from $S_f = S$ to be

$$v_0 = 2.38 \text{ mph} ;$$

the propagation speed of small disturbances corresponding to the depth of 20 ft. is

$$c_0 = \sqrt{gy_0} = 17.3 \text{ mph} .$$

The problem then is to determine the solution $v(x,t)$, $c(x,t)$ for all later time, $t \geq 0$, along the river, $x \geq 0$. Figures 1 and 2 present the result of the computation in the form of stage and discharge curves plotted as functions of distance along the river at various times. As stated above, we worked with the mile as the unit of length, and the hour as the unit of time. Our stage measurements were then converted into the customary unit, the foot, and the discharge into cubic feet per second.

2.2 Computational techniques. In order to indicate how the solution was calculated it is convenient to refer to a diagram in the (x,t) plane (also see Figure 3).

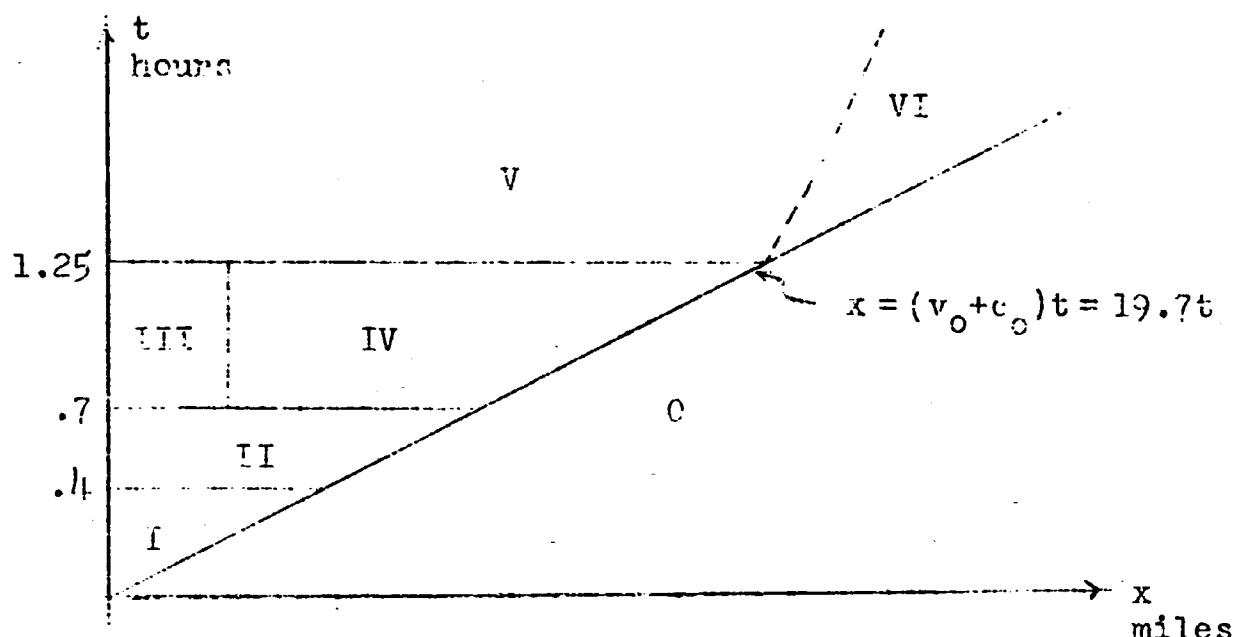


Diagram 1

Regions in which various computational methods were tried

According to the theory presented in Report I, we know that for $x \geq (v_0 + c_0)t = 19.7 t$, called region 0, the solution is given by the unchanged initial data, $v(x,t) = v_0$, $c(x,t) = c_0$ (since the forerunner of the disturbance travels at the speed $v_0 + c_0 = 19.7$ mph).

In order to compute the solution by finite difference methods, we found it necessary to use rather small intervals near the origin. This is to be expected since the values of c_t were changed discontinuously at $t = 0$, $x = 0$. We experimented with various interval sizes and finite difference schemes in order to determine the least time consuming way to calculate the progress of the flood. Our conclusions have been summarized in the introduction to this report. We shall now describe the various schemes and the regions in which they were used (see Diagram 1 and Figure 3).

Region I, $0 \leq x \leq 19.7 t$, $0 \leq t \leq .4$. Intervals $\Delta x = 1$ mile and $\Delta t = .043$ hour were required owing to the sudden increase of depth at $x = 0$, $t = 0$. The finite difference formulae derived in Report I, eqs. (4.8), (4.9) were used:

$$\begin{aligned}
 (2.2) \quad v_P &= v_M + \frac{\Delta t}{\Delta x} \left[(c_M + v_M) \left(\frac{1}{2} v_L - \frac{1}{2} v_M + c_L - c_M \right) \right. \\
 &\quad \left. - (c_M - v_M) \left(\frac{1}{2} v_M - \frac{1}{2} v_R - c_M + c_R \right) - \Delta x E_M \right], \\
 c_P &= c_M + \frac{1}{2} \frac{\Delta t}{\Delta x} \left[(c_M + v_M) \left(\frac{1}{2} v_L - \frac{1}{2} v_M + c_L - c_M \right) \right. \\
 &\quad \left. + (c_M - v_M) \left(\frac{1}{2} v_M - \frac{1}{2} v_R - c_M + c_R \right) \right].
 \end{aligned}$$

where P refers to a point Δt later than M, while L and R are Δx units to the left and right of M respectively.

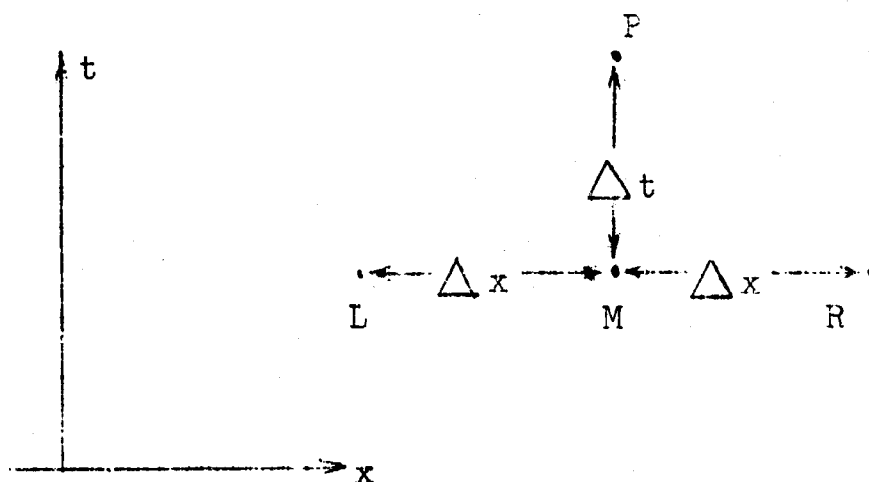


Diagram 2

Net points used in rectangular scheme

(We have adopted the convention of using v_M to denote the value of the function v at the point M.)

In Region II, $0 \leq x \leq 19.7 t$, $.4 \leq t \leq .7$, with $\Delta x = 1$ mile, $\Delta t = .024$, we experimented with a different finite difference scheme, called a "staggered" scheme. This method is appropriate for use when the space intervals are uniform in size along the length of the river. The scheme is described in Report I, eq. (4.14) and eliminates any reference to values at the point M.

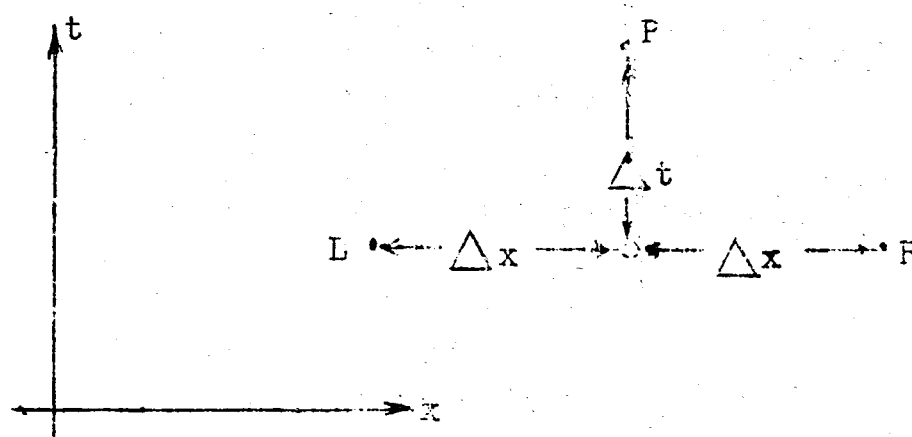


Diagram 3

Net points used in "staggered" scheme

$$v_P = v_{M*} + \frac{\Delta t}{\Delta x} \left[2(c_L - c_R)c_{M*} + (v_L - v_R)v_{M*} - \Delta x E_{M*} \right] \quad (2.3)$$

$$c_P = c_{M*} + \frac{1}{2} \frac{\Delta t}{\Delta x} \left[2(c_L - c_R)v_{M*} + (v_L - v_R)c_{M*} \right]$$

where

$$v_{M*} = \frac{1}{2}(v_L + v_R), \quad c_{M*} = \frac{1}{2}(c_L + c_R), \quad E_{M*} = \frac{1}{2}(E_L + E_R).$$

We must pay a slight penalty for using this somewhat simpler formula. That is, we find that in order to calculate the value of $v(0,t)$, the velocity at the head of the river, we must introduce some additional consideration. We refer to

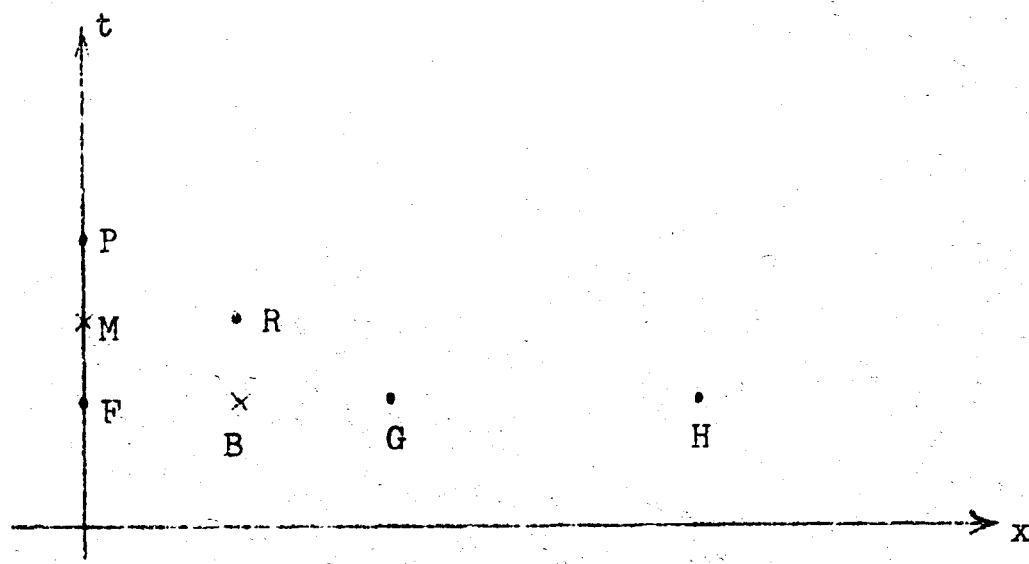


Diagram 4

Net point arrangement used at boundary in "staggered" scheme

Diagram 4, where we indicate schematically the staggered net points P, R, F, G and H. It is clear that equations (2.3) together with the prescribed values of $c(0,t)$ do not produce a formula for v_P in terms of quantities evaluated at the preceding staggered points R, F, G, H, etc. We can overcome this difficulty by determining values for v_B , c_B by linear interpolation* from the values at the points F and G (or if rapid changes are made in the boundary values, we may use quadratic** interpolation on the values at the three points F, G, H). That is, we use such determined values of v_B , c_B in

* For linear interpolation

$$v_B = \frac{1}{2}(v_F + v_G) \quad , \quad c_B = \frac{1}{2}(c_F + c_G) \quad .$$

** For quadratic interpolation

$$c_B = (3c_F + 6c_G - c_H)/8 \quad , \quad v_B = (3v_F + 6v_G - v_H)/8 \quad ,$$

we used these formulae to smooth the data near the boundary when we changed interval sizes.

the difference equation (2.2) to determine the value of v_M . We then use (2.2) again to determine v_P . The necessity for using a different formula to calculate at boundary points would add some difficulty to a scheme for automatic computation.

Region III, $0 \leq x \leq 5$ miles, $.7 \leq t \leq 1.25$ hr, with $\Delta x = 1$ mile, $\Delta t = .024$ hr. We used the same formulae as in Region II (except for points on the boundary between Regions III and IV, see below).

Region IV, $5 \leq x \leq 19.7 t$, $.7 \leq t \leq 1.25$, with $\Delta x = 2$ miles, $\Delta t = .048$ hr. The values at the boundary between Regions III and IV were obtained by linear interpolation from the neighboring values. Other quantities were computed by the "staggered" scheme as in Regions II and III.

Region V, $0 \leq x \leq Vt$, $1.25 \leq t \leq 10$ hrs, $\Delta x = 5$ miles, $\Delta t = .17$ hr. V represents a variable speed which marks the downstream end of the observable disturbance ($V \approx 10$). That is, by using an expansion scheme (see Appendix I) we determined a

Region VI, $Vt \leq x \leq 19.7 t$, about the forerunner of the disturbance, in which the flow is practically undisturbed to the accuracy we were interested in. We therefore used this expansion to calculate the various quantities in Region VI and used the staggered scheme to compute the values in Region V. We permitted Regions V and VI to overlap slightly, in order that (a) we should avoid having to interpolate for boundary values and (b) that we might have a check on the consistency of the two methods of computation.

2.3 Observations. The conclusions reached on the basis of these calculations of a flood in the Ohio can be summarized as follows:

- (a) The model flood, with a 5 foot per hour rise, is extreme even in comparison with the 1945 flood for which the rate of rise at Wheeling, West Virginia was less than .7 foot per hour. Such a case exaggerates the way in which our finite difference methods propagate small errors. For example, we found slight inaccuracies developing at the head, $x = 0$, when we increased the Δx interval size. We could have devised special techniques to smooth this effect--but we didn't in order that we could better observe how the errors were propagated.
- (b) In spite of the exceptionally high rate of rise, the inaccuracies created by using our finite difference methods were damped out rather strongly (in about 8-10 time steps.). It is possible to control these inaccuracies by using appropriate smoothing techniques or more simply by using small interval sizes.
- (c) The process by which the small errors of the finite difference scheme die out may be described as follows: A larger value of v produces a larger friction force which slows down the motion and produces at the next time a smaller velocity. The lower velocity in a similar way then operates through the resistance to create a larger velocity and the process repeats with a steady decrease in the amplitude of variation.

- (d) We checked the accuracy of our computation (as a function of the interval size) by repeating the calculation for various interval sizes over the same region in space and time. Such self-checking by changing intervals can be coded into an automatic procedure for computation.
- (e) The "staggered" scheme introduced additional difficulties for the computation of boundary values (in contrast with the rectangular schemes). This trouble was emphasized in the computation at the boundary of Regions III and IV (smaller Δx intervals were used in Region III). Our plan to use the stage gaging stations as net points would be simpler to code with a rectangular scheme; and would not require any special treatment at the boundaries.
- (f) We have yet to investigate the possibility of using "implicit" finite difference methods--which may permit the use of larger time steps--at the extra cost of using more complicated procedures to solve the equations at each time step.
- (g) The linearized theory, obtained for a small perturbation about the uniform flow with 20 ft. depth, did not give an accurate description of the solution of our problem. We compared the stage profiles predicted by the linear theory and those calculated from the non-linear equations. After a period of less than 2 hours, there was a deviation of 2 feet. The linear theory predicted smaller values for stage and discharge.

- (h) It would be convenient to be able to obtain in advance a safe estimate for the maximum value of the particle velocity, in order to select an appropriate safe value for Δt , since we must have $\Delta t \leq \frac{\Delta x}{v+c}$ (see Report I, p. 26). Our calculations indicate that this may not be too simple theoretically, for the maximum velocity at $x = 0$ greatly exceeds its asymptotic value.

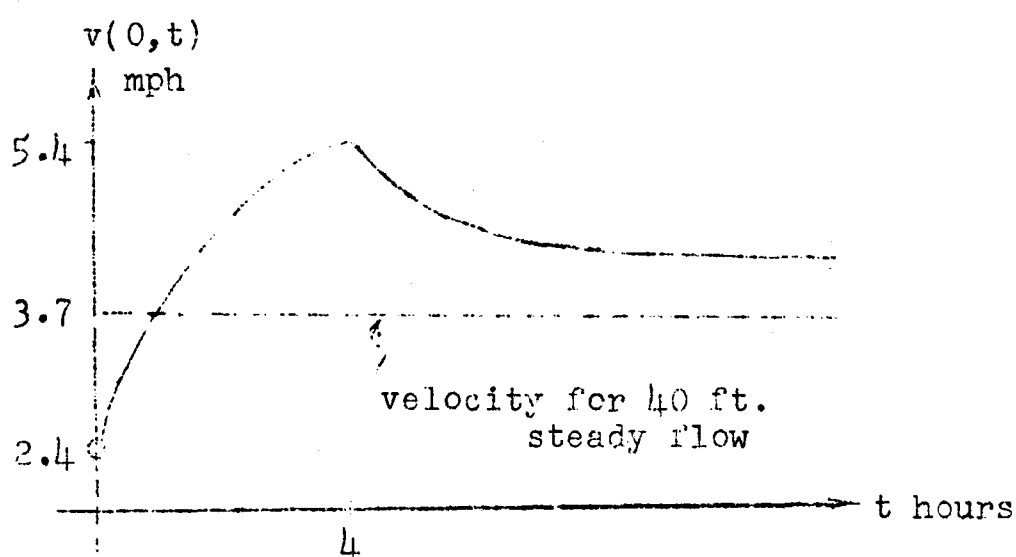


Diagram 5

Water velocity obtained at "head" of
river through regulation of stage

But, it may well be possible, on the basis of empirical observations, to obtain adequate estimates of the maximum velocity. As was indicated earlier, we can easily code the calculation of the proper interval size as a part of the computation should other means of determination be inadequate.

- (1) We note that the curves of constant stage have slopes which are closer to 5 mph (the speed with which a steady progressing flow, 40 ft. upstream and 20 ft. downstream, moves, see next section) than to the over 19 mph speed of propagation of small disturbances.

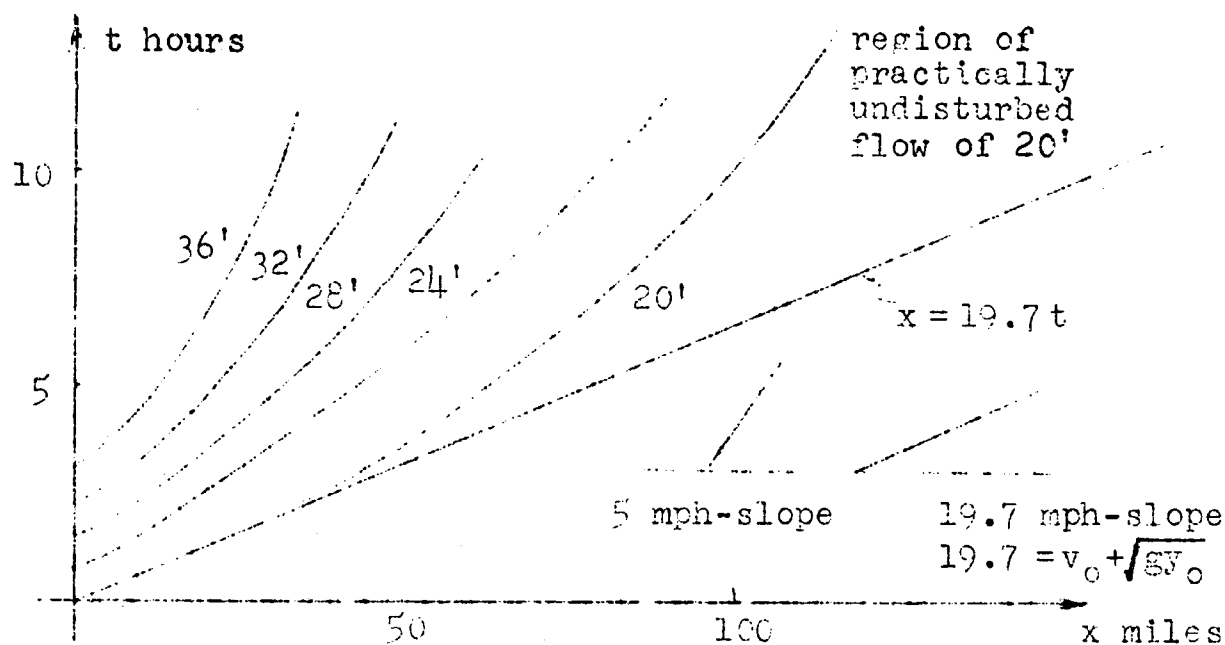


Diagram 6

Curves of constant stage--comparison with first characteristic and steady progressing flow velocity

The region of practically undisturbed flow (determined by expansion about the "first" characteristic $x = 19.7 t$, see Appendix I) is shown above. In practice, we expect the local runoff discharges and the non-uniform flow conditions to eliminate the region of practically undisturbed flow. For this reason, in general, we can not use analytic expansion schemes to try to save computational labor of this kind.

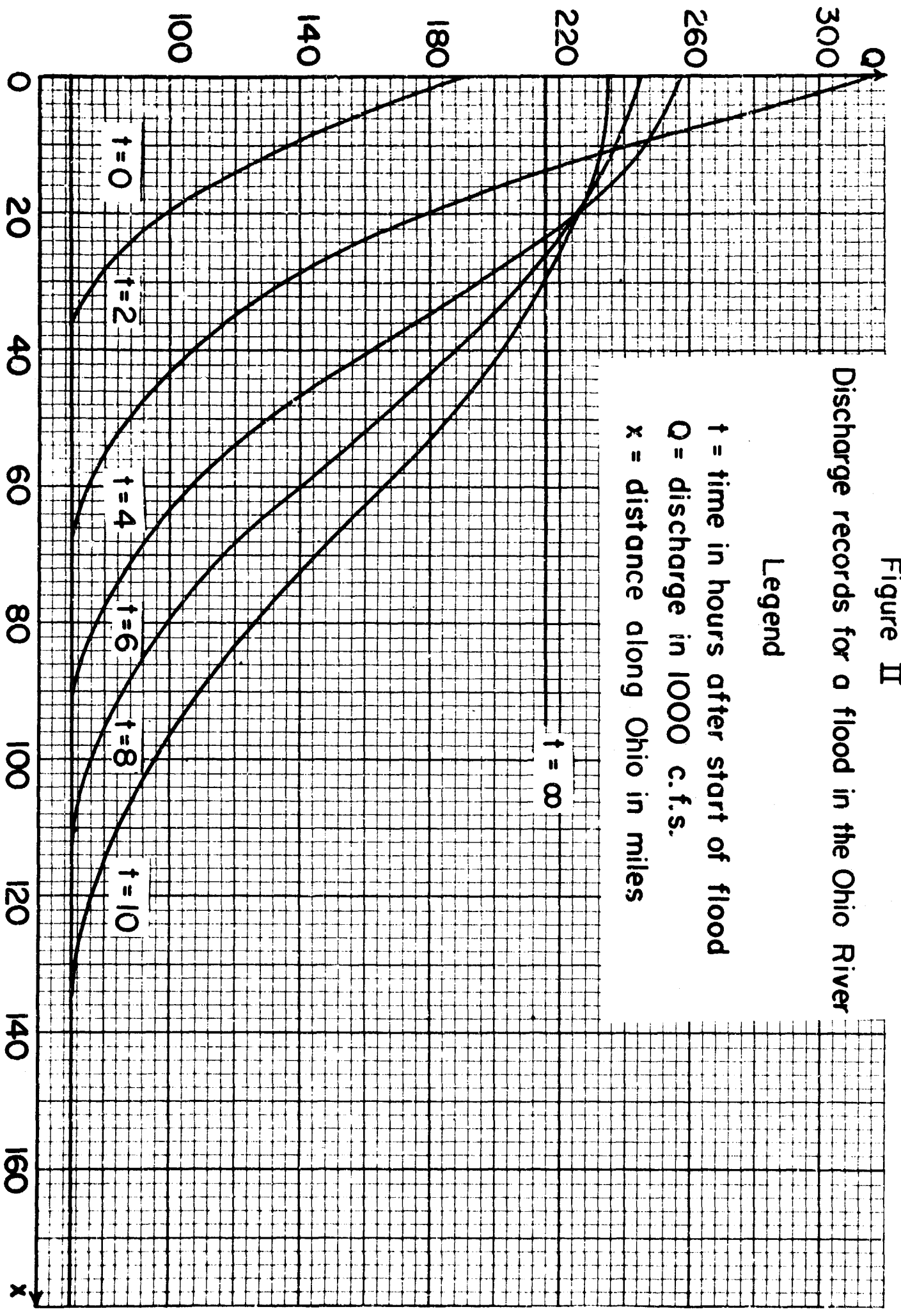


Figure I

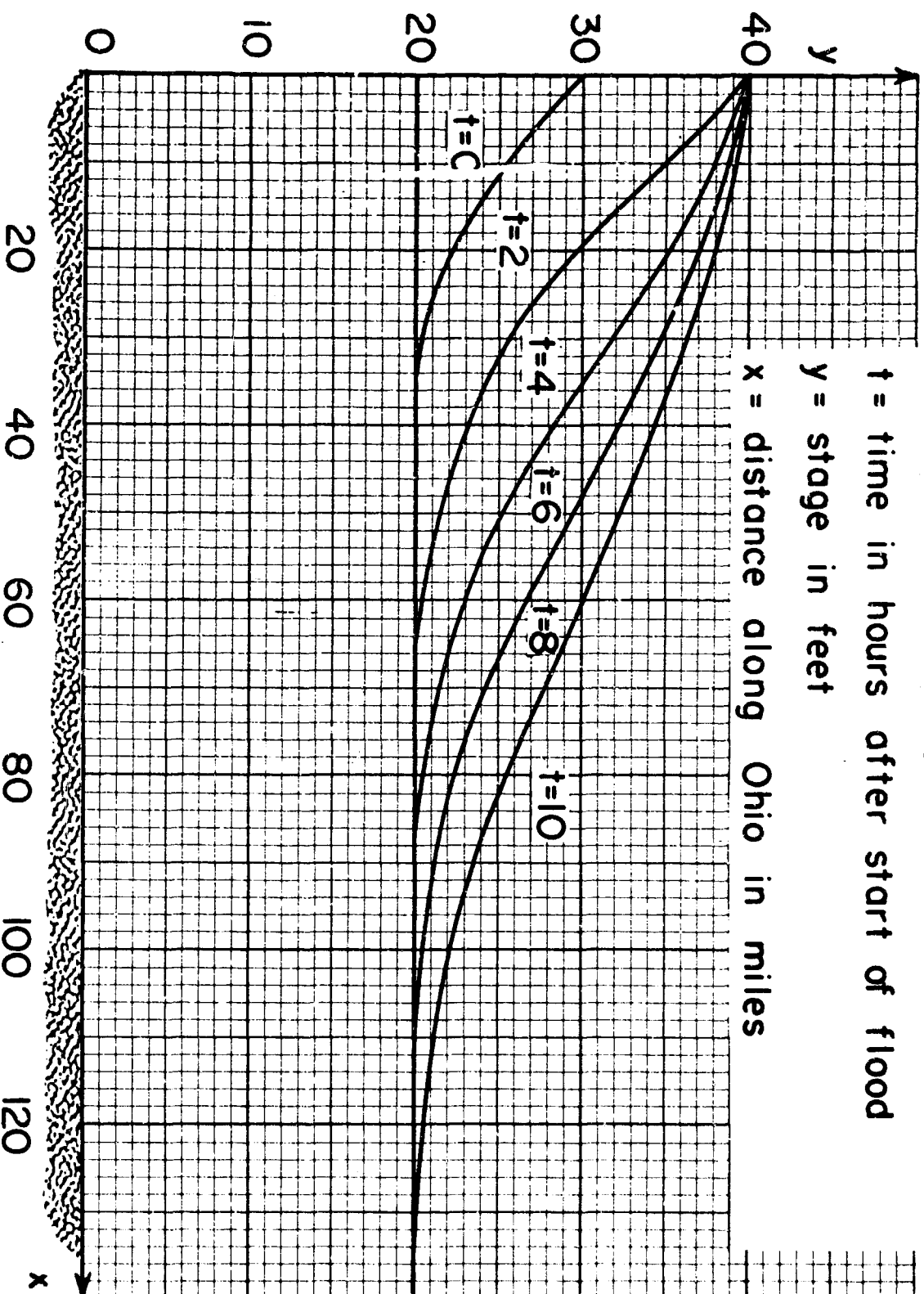
Stage profiles for a flood in the Ohio River

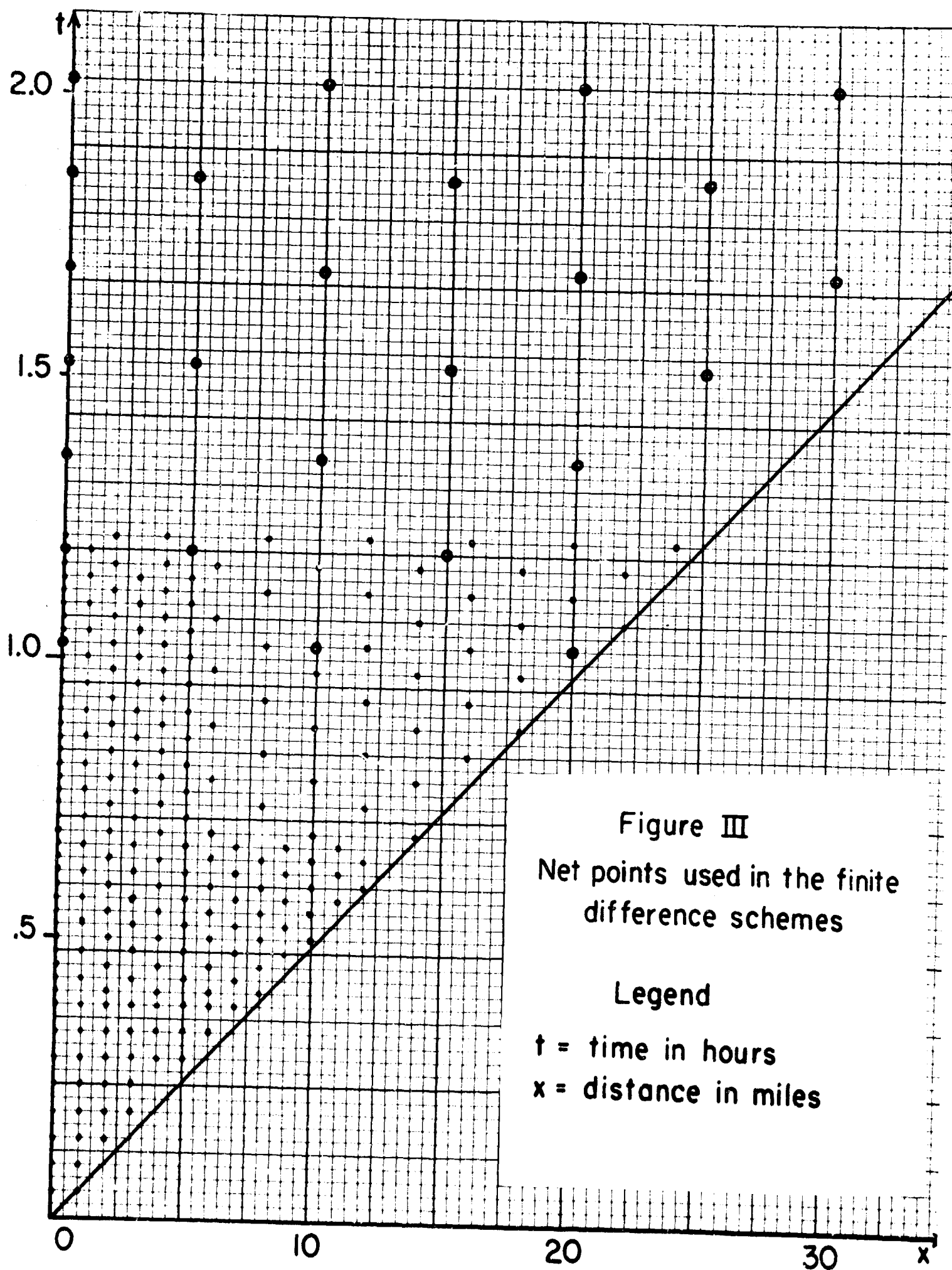
Legend

t = time in hours after start of flood

y = stage in feet

x = distance along Ohio in miles





2.4 Determination of accuracy of finite difference method by comparison with an exact solution. There is a case in which an exact solution of the differential equations is known, i.e. the case of a steady progressing wave* with two different depths at great distances upstream and downstream. This exact solution for the case of a wave of depth 20 ft. far downstream and 40 ft. far upstream was taken as furnishing the initial conditions at $t = 0$ for a wave motion in the river. With the initial conditions prescribed in this way the finite difference method was used to determine the motion at later times; of course the calculation, if accurate, should furnish a wave profile and velocity distribution which is the same at time t as at the initial instant $t = 0$ except that all quantities are displaced downstream a distance Ut , with U the speed of the steady progressing wave. In this way an opportunity arises to compare the approximate solution with an exact solution. In the present case the phase velocity U is approximately 5 mph. We considered an interval size of $\Delta x = 10$ miles in a "staggered" scheme with $\Delta t = .34$ hr. After 7 hours, the calculated stage values agreed to within .6 per cent with the exact values. The discharge and the velocity deviated by less than 2.5 per cent from the exact value. The fact that stage values can be predicted more

* In Appendix II we derive the formulae which describe the steady progressing wave.

accurately than discharge values is a characteristic of the presently used finite difference methods that was observed in all calculations.

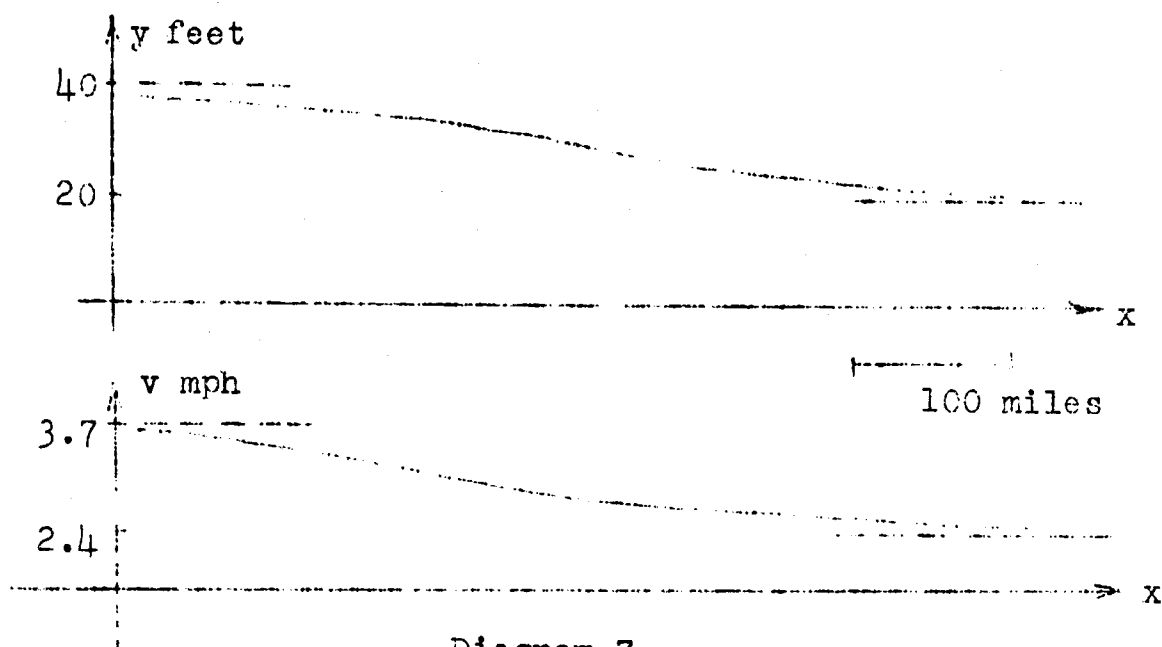


Diagram 7

Steady progressing flow--stage and velocity profiles

3. The junction of the Ohio and the Mississippi

3.1 Formulation of the junction problem. Our model for the junction of the Ohio and Mississippi Rivers was constructed to test our finite difference methods with regard to their applicability in such cases. We believe that the consideration of the actual problem, as it occurs in nature, would present no new difficulties in principle (except for the consideration of overbank flow). We supposed the upstream Mississippi section to be identical with the Ohio River--rectangular cross-section, 1000 ft. wide, slope of .5 ft./mile, Manning's $n=.03$, infinite in length. We imagined the downstream Mississippi branch to be rectangular, twice as wide, i.e. 2000 ft.,

Manning's $n = .03$, and a slightly smaller slope .49 ft./mile. The latter modification of the slope was made in order to obtain an initial solution corresponding to a uniform flow of 20 ft. depth in all three branches (such a change is necessary to overcome the decrease in wetted perimeter which occurs on going downstream through the junction).

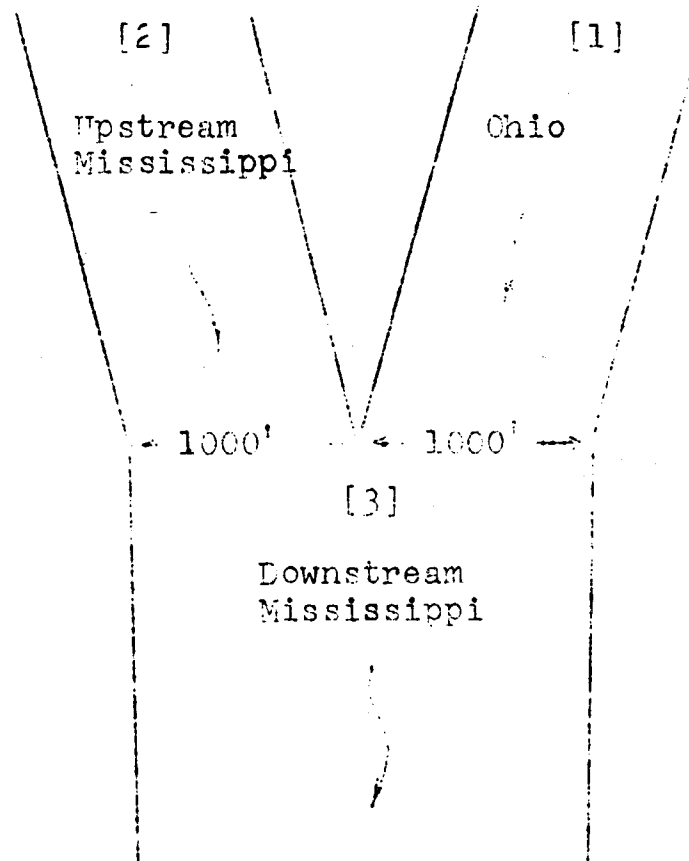


Diagram 8

Schematic plan of junction

We considered the problem of starting a flood wave in the Ohio, in fact, the same flood we had computed earlier. The difference was that we began the Ohio flood 50 miles upstream from the junction. After about 2.5 hours the forerunners of the Ohio disturbance reached the junction. They then separated into two wavelets, one going upstream and the other

downstream on the Mississippi. In addition, at the same instant, a reflected wavelet started backwards up the Ohio. Our finite difference calculations were begun in all three branches from the moment that the junction was reached by the forerunner of the Ohio flood. (We saved the labor of recomputing the motion for the first 2.5 hours in the Ohio, since the motion was identical with the one previously calculated.) The flow was followed for 10 hours and the results are indicated in the stage and discharge plots as a function of distance (for various times) given in Figures 4 and 5.

3.2 Description of the computational procedures. Let $v[1]$, $c[1]$, $v[2]$, $c[2]$, $v[3]$, $c[3]$ represent the velocity v and propagation speed c for the Ohio, upstream Mississippi, and downstream Mississippi, respectively. A "staggered" scheme was used with interval $\Delta x = 5$ miles and $\Delta t = .17$ hr. as indicated in Diagram 9. The junction point is denoted by $x = 0$, the region of the Ohio and the upstream Mississippi are represented by $x \leq 0$, while the downstream Mississippi is given on $x \geq 0$. The time $t = 2.5$ hr. corresponds to the instant that the forerunner of the flood reaches the junction.

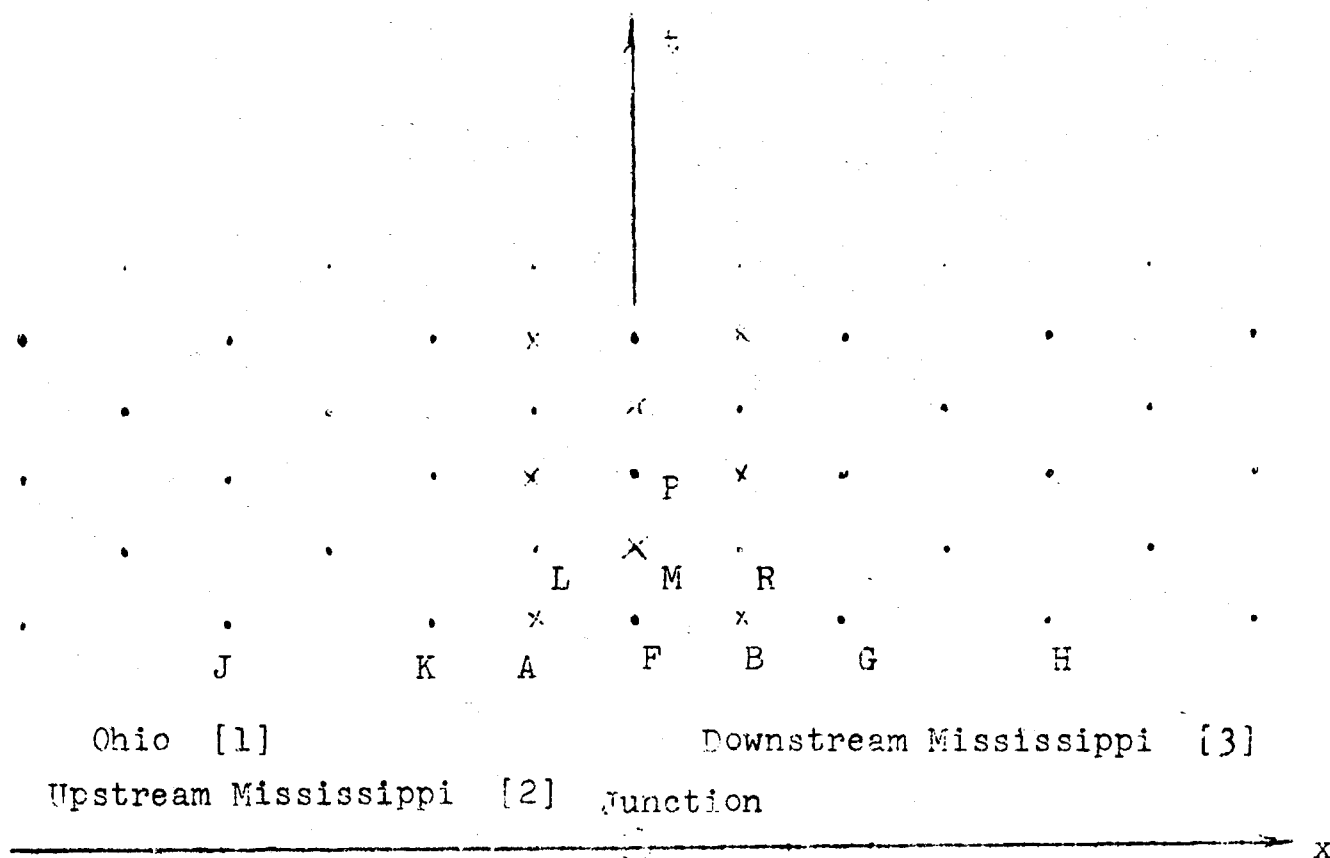


Diagram 9

Junction net point scheme

The values of the quantities v and c at the junction were determined as follows: Let us assume that the values of v and c have been obtained at all net points for times preceding that of the boundary net point P . We know that at the junction

$$c_{[1]} = c_{[2]} = c_{[3]}$$

(since $c = \sqrt{gy}$ and the three branches are level at the junction) and

$$y_{[1]}v_{[1]} + y_{[2]}v_{[2]} = 2y_{[3]}v_{[3]}$$

(since what flows into the junction from the upstream Mississippi and the Ohio must flow out of the junction into the downstream Mississippi). If the values of v and c were known at M in the respective branches of the rivers, we could find the values at P from the following three equations: Report I, eq. (4.6) for the Ohio and upstream Mississippi, and Report I, eq. (4.7) for the downstream Mississippi. We rewrite all of the equations for convenience

$$c_{P[1]} = c_{P[2]} = c_{P[3]} \quad (\text{with } c = \sqrt{gy})$$

$$v_{P[1]} + v_{P[2]} = 2v_{P[3]} \quad (\text{since } y_{[1]} = y_{[2]} = y_{[3]})$$

$$(3.1_{[1]}) \quad 2 \left\{ \frac{c_{P[1]} - c_{M[1]}}{\Delta t} + (c_{M[1]} + v_{M[1]}) \frac{(c_{M[1]} - c_{L[1]})}{\Delta x} \right\} \\ + \left\{ \frac{v_{P[1]} - v_{M[1]}}{\Delta t} + (c_{M[1]} + v_{M[1]}) \left(\frac{v_{M[1]} - v_{L[1]}}{\Delta x} \right) \right\} + E_{M[1]} = 0,$$

$$(3.1_{[2]}) \quad 2 \left\{ \frac{c_{P[2]} - c_{M[2]}}{\Delta t} + (c_{M[2]} + v_{M[2]}) \frac{(c_{M[2]} - c_{L[2]})}{\Delta x} \right\} \\ + \left\{ \frac{v_{P[2]} - v_{M[2]}}{\Delta t} + (c_{M[2]} + v_{M[2]}) \left(\frac{v_{M[2]} - v_{L[2]}}{\Delta x} \right) \right\} + E_{M[2]} = 0,$$

and

$$(3.2_{[3]}) \quad -2 \left\{ \frac{c_{P[3]} - c_{M[3]}}{\Delta t} + (v_{M[3]} - c_{M[3]}) \frac{c_{P[3]} - c_{M[3]}}{\Delta x} \right\} \\ + \left\{ \frac{v_{P[3]} - v_{M[3]}}{\Delta t} + (v_{M[3]} - c_{M[3]}) \frac{v_{P[3]} - v_{M[3]}}{\Delta x} \right\} + E_{M[3]} = 0.$$

The above system of six equations determines uniquely the values $v[1]$, $c[1]$, $v[2]$, $c[2]$, $v[3]$, $c[3]$ at P in terms of their values at the preceding points L, M and R. The equations can be solved explicitly. On the other hand, the values at M are determined in the same way from the preceding values at A, F and B. The values at A and B are determined by linear* interpolation between the neighboring points (K,F) and (F,G) respectively (see Diagram 9).

3.3 Observations.

- (a) We find that the flow tends to a steady state in which the downstream Mississippi has uniform flow, as indicated below (see Appendix II).

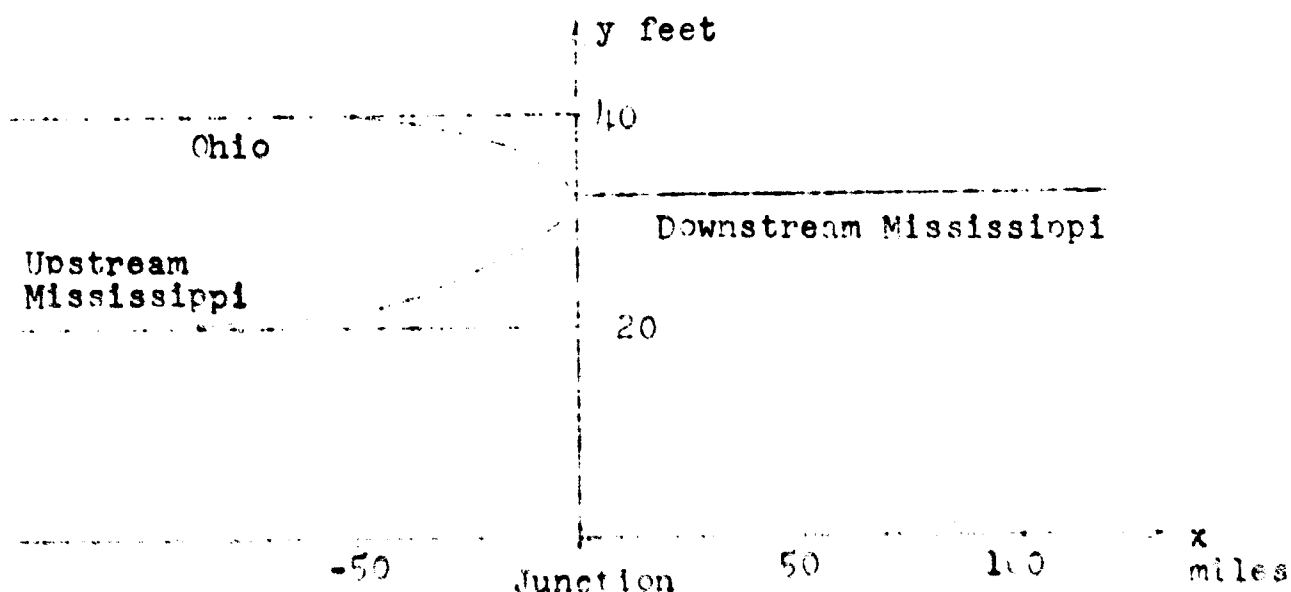


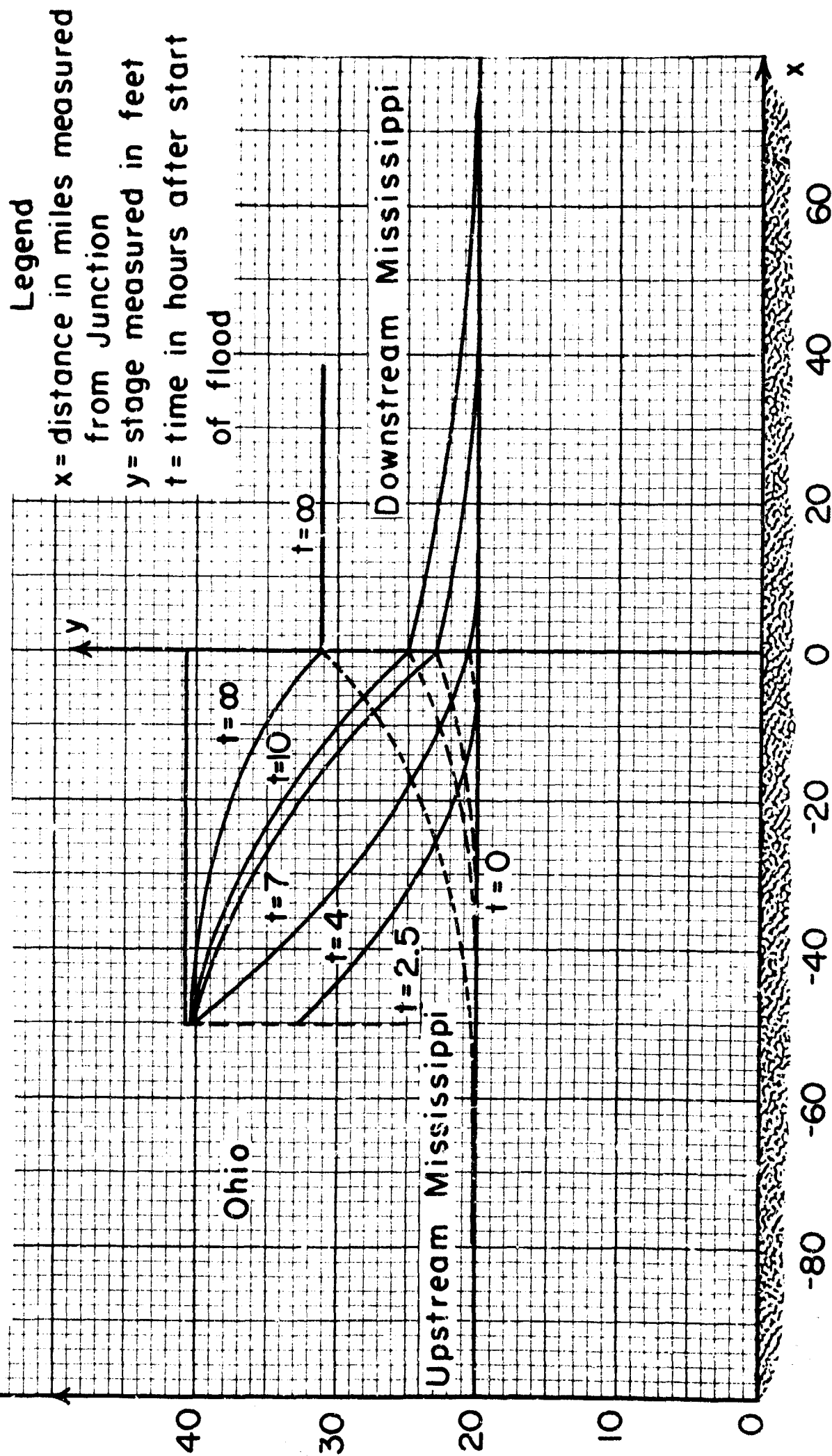
Diagram 10

Asymptotic flow in junction problem

* Quadratic interpolation using (F,G,H) and (J,K,F) was used after the stage increased appreciably at the junction.

Figure IV

Stage profiles for a flood at the Junction of the
Ohio and Mississippi Rivers



- (b) The existence of both an up- and downstream branch of the river into which the Ohio discharges has the effect of lowering the Ohio River level. A crude estimate of what would happen should the Ohio discharge only into the downstream branch of the Mississippi, indicates a rise of about 5 per cent in the level of the Ohio at the junction.
- (c) There is a definite influence on the upstream branch of the Mississippi (see Figures 4 and 5) and it is in this connection that the finite difference methods display their advantage over the standard flood routing procedures. For example, at a point 50 miles upstream in the Mississippi the stage is increased by 1 ft. as a result of the pouring of the Ohio River flood into the junction.
- (d) We found it convenient to compute the flow in the neighborhood of the flood forerunners in the two branches of the Mississippi by means of the expansion scheme mentioned earlier and described in detail in Appendix I.
- (e) The damping of small errors and other features peculiar to these equations are the same in the present case as in the preceding case of the Ohio; they are described in section 2.3 (in connection with the Ohio River model flood calculation).

Appendix I

Expansion in the neighborhood of the first characteristic.

It has been mentioned already that, whereas the first signal of a disturbance initiated at a certain point in a river with uniform flow travels downstream with the speed $v + \sqrt{gy}$, the main part of the flood wave travels more slowly (cf. Deymié*), depending strongly on the resistance of the river bed. An investigation of the motion near the head of the wave, i.e. near the first characteristic with the equation $x = t(v_0 + c_0)$, shows immediately why the main part of the disturbance will in general, but not always, fall back behind the forerunner.

The motion is investigated in this Appendix by means of an expansion of G. Whitham and A. Troesch and is carried out to the two first orders for the model of the Ohio River, and to the lowest order in the much more complicated case of the junction problem. The results obtained enable us to improve the accuracy of the solution near the first characteristic. It turns out that the finite difference scheme yields river depths which are too big (see below), owing to the fact that our large mesh width smoothed the river profile out.

* Deymié, Ph. Propagation d'une intumescence allongée (problème aval). Int. Cong. for Appl. Mech., 5th. Proceedings, Sept. 1938, N. Y., Wiley, 1939.

Figure V

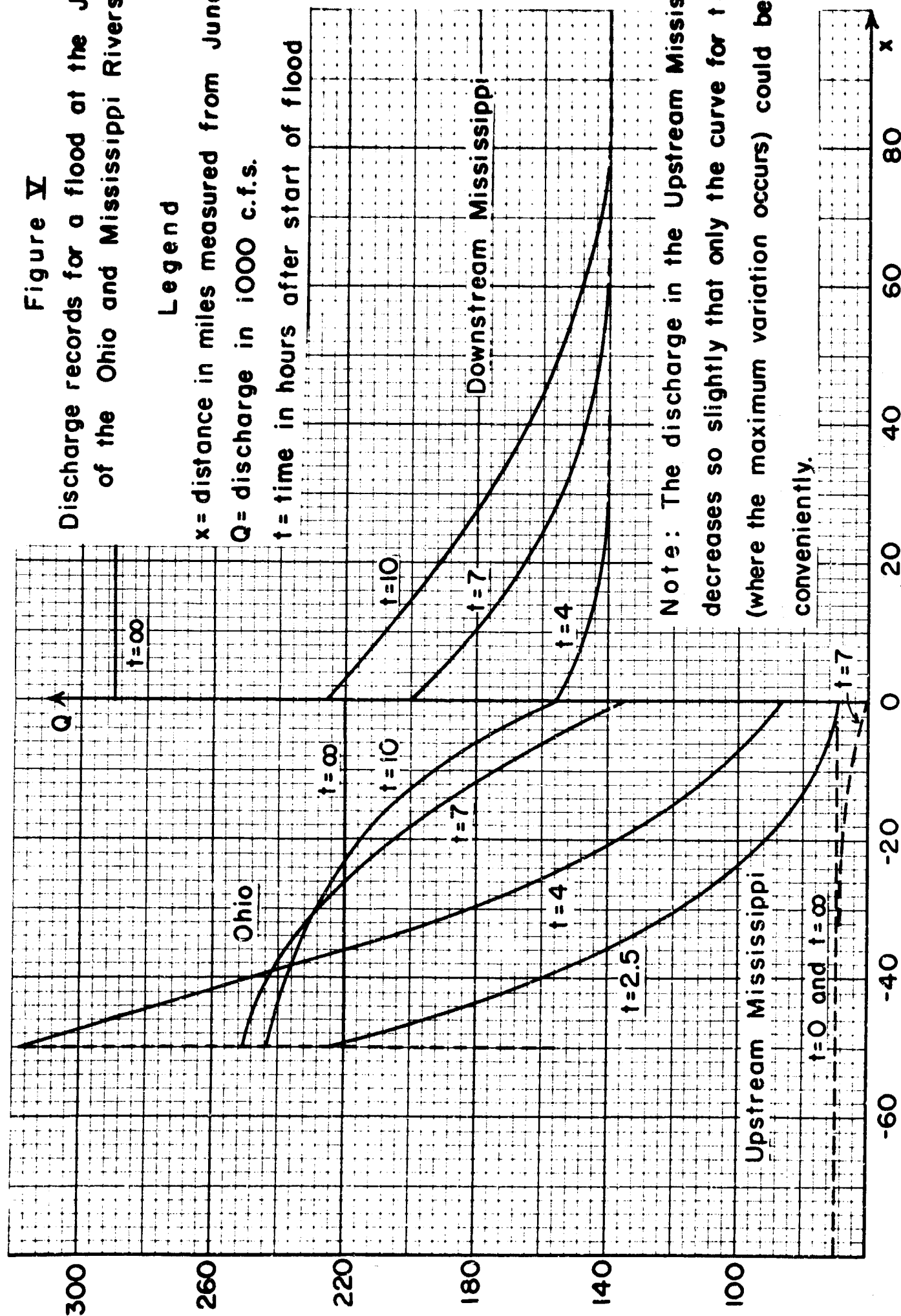
Discharge records for a flood at the Junction
of the Ohio and Mississippi Rivers

Legend

x = distance in miles measured from Junction

Q = discharge in 1000 c.f.s.

t = time in hours after start of flood



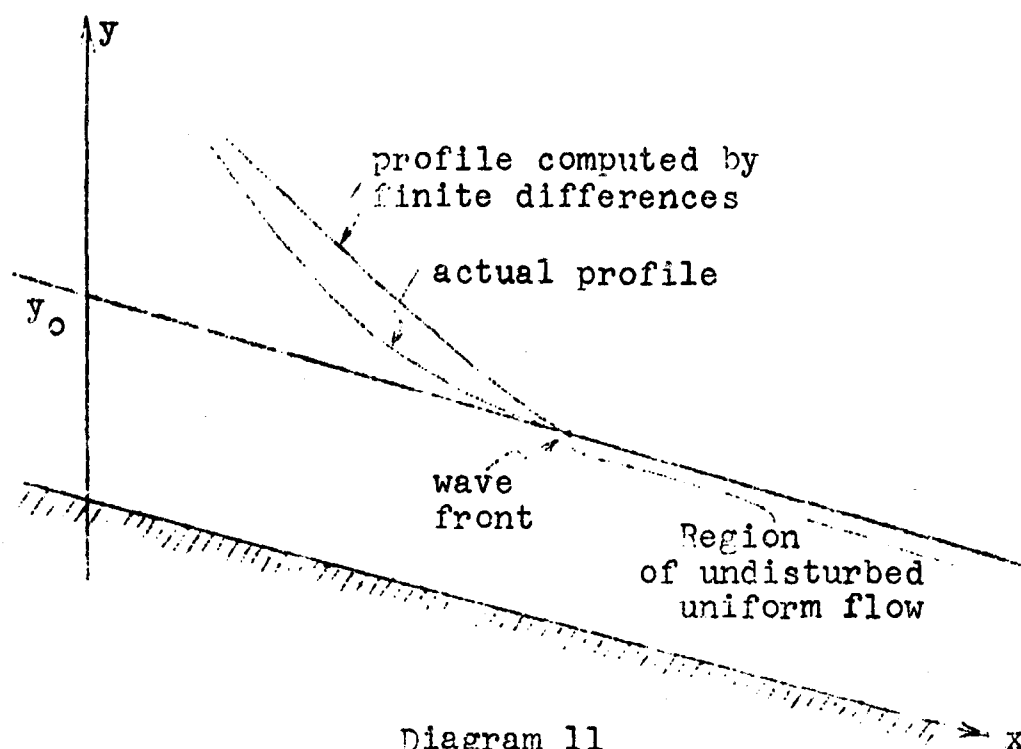


Diagram 11

Error introduced by finite difference scheme
in neighborhood of first characteristic of a
rapidly rising flood wave

In order to expand the solution in the neighborhood of the wave front, we introduce new coordinates

$$\xi = x \quad \text{and} \quad \tau = (v_0 + c_0)t - x$$

such that the ξ -axis (i.e. $\tau = 0$) coincides with the first characteristic. The basic system of equations (2.1) are restated for convenience

$$(A.1) \quad \begin{aligned} 2cc_x + v_t + vv_x - gS + gS_f &= 0, \\ cv_x + 2vc_x + 2c_t &= 0. \end{aligned}$$

Upon substitution of the new variables ξ and τ we find

$$2c(c_\xi - c_\tau) + v(v_\xi - v_\tau) + (v_0 + c_0)v_\tau - gS + gS_f = 0 \quad ,$$

$$2v(c_\xi - c_\tau) + c(v_\xi - v_\tau) + 2(v_0 + c_0)c_\tau = 0 \quad ,$$

where the friction slope S_f for a rectangular channel of width B is given by

$$S_f = \frac{v|v|}{\gamma \left(\frac{y}{1 + \frac{2y}{B}} \right)^{4/3}} = \frac{g^{4/3}}{\gamma} v|v| \left\{ \frac{1}{c^2} + \frac{2}{gB} \right\}^{4/3} .$$

We expand v and c as power series in τ with coefficients that are functions of ξ as follows:

$$v = v_0 + v_1(\xi)\tau + v_2(\xi)\tau^2 + \dots \quad ,$$

$$c = c_0 + c_1(\xi)\tau + c_2(\xi)\tau^2 + \dots \quad .$$

This expansion is to be used for $\tau > 0$ only, since for $\tau < 0$ we are in the undisturbed region and all functions $v_1(\xi), v_2(\xi), \dots, c_1(\xi), c_2(\xi), \dots$ vanish identically. If we insert the series for v and c into equations (A.1) and collect terms of the same order in τ , we get ordinary differential equations for $v_1(\xi), c_1(\xi), \dots$ for example

$$4(v_0 + c_0)^2 \cdot \frac{dc_1}{d\xi} - 12c_1^2 + 4c_1 \cdot gS \left\{ \frac{1}{v_0} - \frac{2}{3c_0} \cdot \frac{1}{1 + \frac{2c_0^2}{gB}} \right\} = 0 \quad .$$

Although the solution of this differential equation for $c_1(\cdot)$ is easily obtained, the result expressed in general terms looks so complicated that we prefer to give it only for the

case of the Ohio River under the conditions discussed in section 2: $c_1 = (1.05 + 8.06 e^{0.146\xi})^{-1}$, in miles and hours. This result has the following physical meaning: The angle α

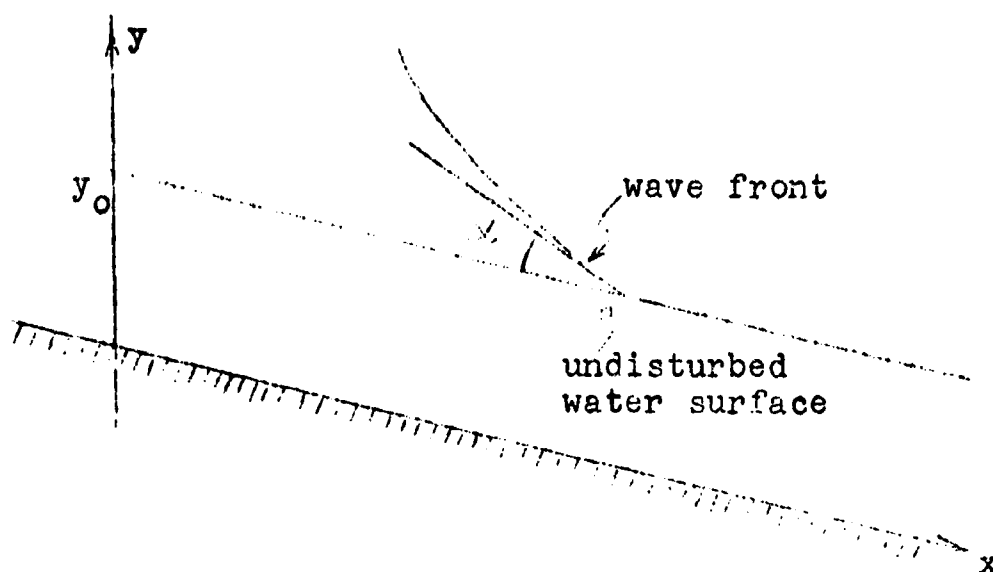


Diagram 12

of the profiles measured between the wave front and the undisturbed water surface is shown to die out exponentially $\alpha \sim \frac{1}{1 + ae^{bx}}$, with a and b constants depending on the river and the boundary condition at $x = 0$. Theoretically, α could also increase exponentially downstream so that a bore would eventually develop, but only if the increase in level at $x = 0$ is extremely fast; in our example (sec. 2.1) no bore will develop unless the water rises at a rate of at least 1 ft. per minute.

Unfortunately, the evaluation of $c_2(\xi)$ which yields the curvature of the profile at the wave front is already very cumbersome. The curvature is found to decrease for large x like xe^{-bx} , b being a positive constant. With the two highest order terms in the expansion known, we were able

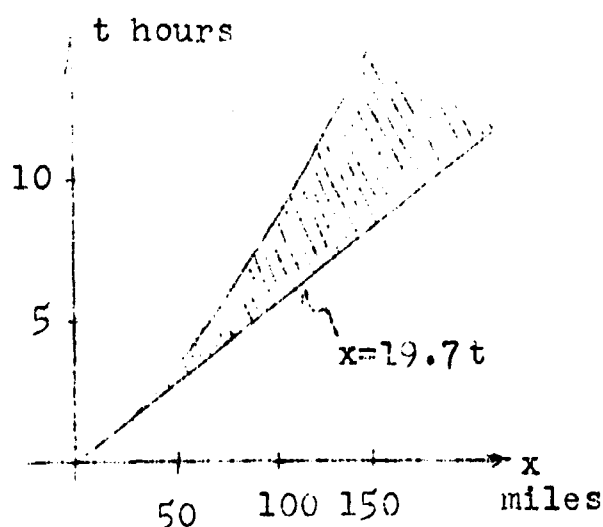


Diagram 13

Region of practically
undisturbed flow

to estimate the region adjacent to the first characteristic where the flow is practically undisturbed. It is remarkable how far behind the theoretical forerunner the first measurable signal travels, see Diagram 13.

In a similar way, an expansion as a power series in ϵ has been carried out

for the junction of the Ohio and Mississippi, as described in section 3. Here even the lowest order term required a complicated computation, since we had to work simultaneously in three different x, t -planes, with boundary conditions at the junction. The differential equations for c_1 are, in all three branches, of the same type as for the Ohio, and their solution for the junction problem as treated in section 3 are $c_1 = .00084 e^{-.145\epsilon}$ for the Mississippi upstream, $c_1 = .00084 e^{-.229\epsilon}$ for the Mississippi downstream, in miles and hours. This means that the angle α dies out exponentially again in the Mississippi, a little faster downstream than upstream, as could be expected, since the oncoming water in the upstream branch makes the wave front steeper.

In the problem of the idealized Ohio River and of the junction the expansions were carried out numerically in full detail and were used to avoid computation by finite differences in a region of practically undisturbed flow.* At the same time the accuracy in the region of small disturbance was improved in general (as indicated above).

* This would become more and more important if the flow were to be computed beyond 10 hours.

Appendix II

Steady flows and steady progressing waves.

II.1 Steady flows. We define a steady flow to be one for which the velocity v and depth y (or, equivalently the propagation speed $c = \sqrt{gy}$) are independent of the time, that is, $v_t = y_t = 0$. It follows from the equation of continuity (see eq. (3.1) in Report I),

$$y_t + vy_x + yv_x = 0 ,$$

that for steady flow

$$(II.1) \quad (vy)_x = 0 \quad \text{whence} \quad vy = D \quad (D \text{ a constant}) .$$

Similarly, the equation of motion (eq. (3.2) of Report I)

$$v_t + vv_x + gy_x + g(S_f - S) = 0 \quad \text{yields}$$

$$(II.2) \quad vv_x + gy_x + g(S_f - S) = 0 .$$

Since from equation (II.1) it follows that

$$\frac{D}{y} \quad \text{and} \quad v_x = - \frac{D}{y^2} y_x ,$$

equation (II.2) becomes

$$(II.3) \quad \left(g - \frac{D^2}{y^3}\right) y_x + g \left(\frac{D^2}{yy^2 \left(\frac{y}{1 + \frac{Dv}{B}}\right)^{4/3}} - S \right) = 0 .$$

We note that equation (II.3) has the simple solution $y = \text{constant}$, for y satisfying

$$(II.4) \quad \frac{D^2}{\gamma S} = y^2 \left(\frac{y}{1 + \frac{2y}{B}} \right)^{4/3} .$$

This means that we can find a flow of uniform depth and velocity having a given discharge BD (B is width of channel). Conversely, by fixing the depth y we can find the discharge from (II.4) of the corresponding uniform flow.

For a channel with varying physical parameters such as cross-section, friction term, etc. the steady flows provide the well-known backwater curves. In general, we would find non-trivial steady solutions $y = y(x)$ and $v = v(x)$ for the non-uniform channel. The explicit determination of the stage and discharge would be possible by numerical integration of ordinary differential equations.

The analysis of the steady flow through the junction of the Ohio and Mississippi Rivers was made for the conditions described in section 3. That is, at 50 miles upstream in the Ohio the stage was held at 40 ft, while far upstream in the Mississippi the stage was kept at 20 feet. The latter condition implies that the slope y_x should vanish upstream in the Mississippi, or that y and $D = vy$ satisfy equation (II.4). As before, let us use subscripts [1], [2], and [3] to represent quantities evaluated in the Ohio, upstream Mississippi and downstream Mississippi respectively. The conditions that must be satisfied at the junction can be simply written

$$(II.5) \quad y_{[1]} = y_{[2]} = y_{[3]} \quad (= y_J \text{ an unknown})$$

$$(II.6) \quad D_{[1]} + D_{[2]} = 2D_{[3]} , \quad \text{for } x = 0 .$$

Equation (II.3) can now be integrated in each of the three river branches (from the junction point as origin).

$$(II.7) \quad \int_{y_J}^y \frac{\left(g - \frac{D^2}{y^3}\right)}{-E} dy = x$$

with

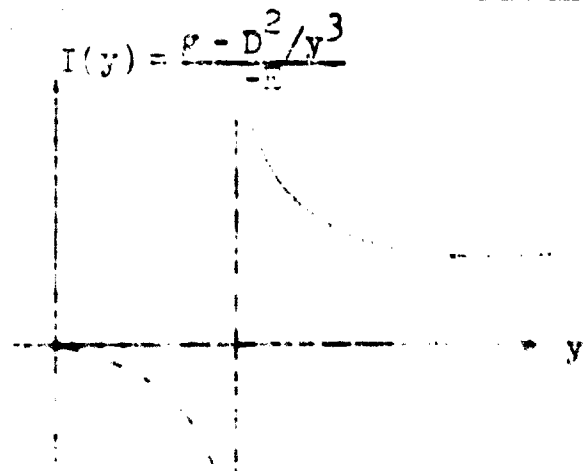
$$E = g \left[\frac{D^2}{\gamma y^2 \left(\frac{y}{1 + \frac{2y}{B}} \right)^{4/3}} - s \right] .$$

We observe that y_J as well as $D_{[1]}$ and $D_{[3]}$ must be determined in order that (II.7) can be used. ($D_{[2]}$ as observed above is known from the upstream condition in the Mississippi.)

The flow in the downstream Mississippi is easily seen* to be uniform (i.e. depth $y = y_J$ and velocity are constant) and this means that equation (II.4) (with the use of (II.6)) holds in the form (if $B = 1000$ ft.)

* We schematically plot the integrand, $I(y)$, of equation (II.7) (for given D and other constants) in the region of sub-critical flow, that is, where gy is greater than v^2 . We see from Diagram 14 that

$$x = \int_{y_J}^y I(y) dy$$



will become positive infinite only if $y = y_J =$ value for

Diagram 14

which $E = 0$. (No other combination of finite values of y and y_J can produce a positive infinite value for x .)

$$(II.8) \quad \frac{(D_{[1]} + D_{[2]})^2}{4y_{S[3]}} = y_J^2 \left(\frac{y_J}{1 + \frac{y_J}{B}} \right)^{4/3}$$

Equation (II.8) together with equation (II.7) rewritten for the 50 mile segment of the Ohio as

$$(II.9) \quad 50 = \int_{y_J}^{\hat{y}_{[1]}} \frac{g - \frac{D_{[1]}^2}{y^3}}{-E} dy \quad \text{with} \quad \hat{y}_{[1]} = \frac{40}{5280} \text{ mi}$$

are a pair of simultaneous equations from which the values of y_J and $D_{[1]}$ can be determined by an iterative procedure.

The results are for $x = 0$,

$$y_{[1]} = y_{[2]} = y_{[3]} = y_J = 31.2 \text{ ft.}$$

$$v_{[1]} = 4.83 \text{ mph}, \quad v_{[2]} = 2.38 \text{ mph}, \quad v_{[3]} = 3.18 \text{ mph}$$

The curve of stage versus distance were computed from formula (II.7) and are drawn below.

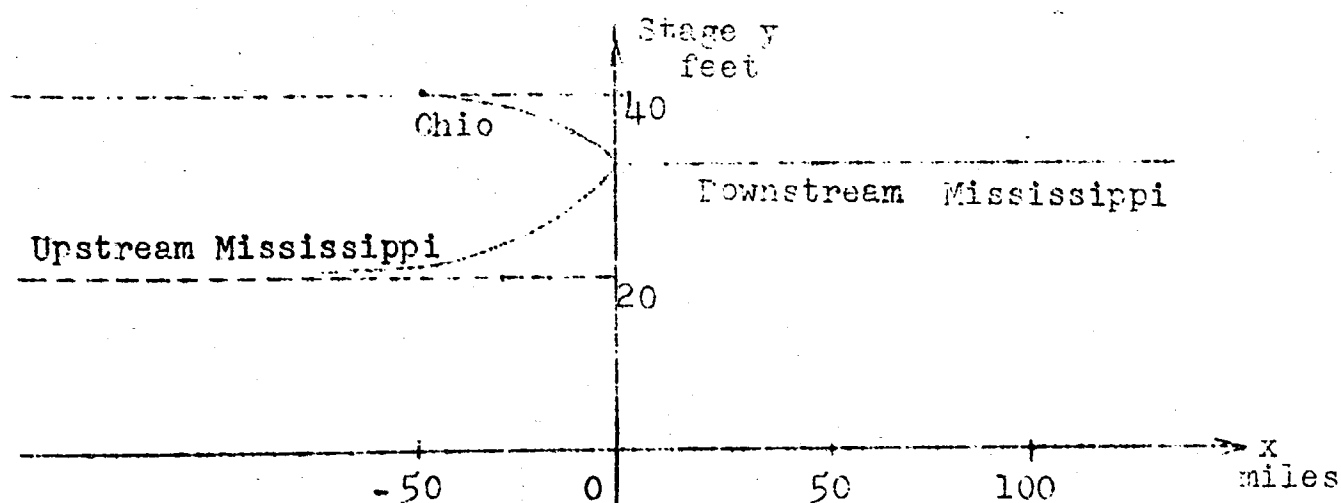


Diagram 15

Steady flow profile in neighborhood of
junction of the Ohio and Mississippi Rivers

We have noted earlier that the transient solution of our junction problem tends to the above drawn steady flow as time increases.

II.2 Steady-progressing waves. For the case of the uniform channel it is possible to find solutions in the form of steady-progressing waves, that is

$$y(x,t) = y(x - Ut) , \quad v(x,t) = v(x - Ut)$$

with U a constant speed of propagation.

In order to find a convenient representation of such phenomena, we introduce a new coordinate system (z,t) by setting

$$z = x - Ut , \quad t = t .$$

We note that $z = \text{constant}$ corresponds to a motion with velocity U in the (x,t) plane. The differential equations become

$$(II.10) \quad y_t + (v - U)y_z + yv_z = 0$$

$$(II.11) \quad v_t + (v - U)v_z + gy_z + g(S_f - S) = 0 .$$

The requirement that y and v be functions of z only, means that $y_t = v_t = 0$ and as in the case of steady flow we find that

$$(v - U)y = D \quad (D \text{ a constant}) \quad \text{and}$$

$$(II.12)$$

$$\left(g - \frac{D^2}{y^3}\right)y_z + g(S_f - S) = 0 .$$

Equation (II.12) can be integrated to yield

$$(II.13) \quad z - z^* = \int_{y^*}^y \frac{\left(g - \frac{D^2}{y^3}\right)}{-E} dy = \int_{y^*}^y I(y) dy \quad .$$

The integrand $I(y)$ for sub-critical flows will be typified in the following sketch.

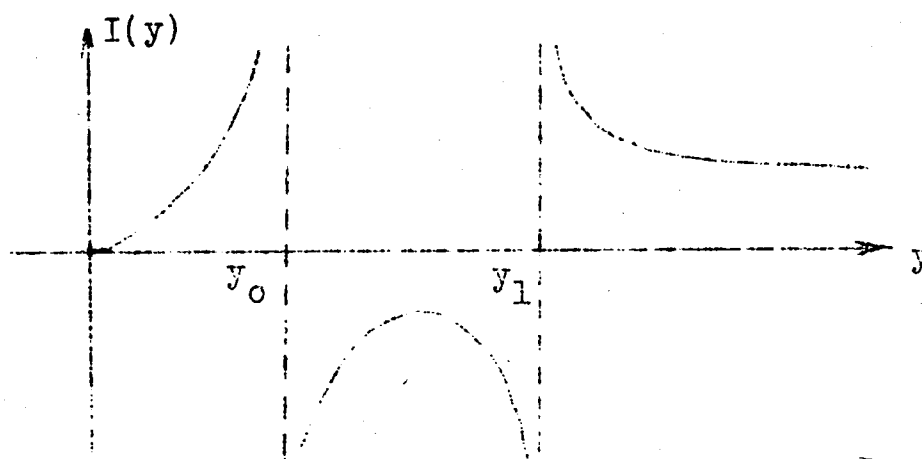


Diagram 16

We observe that there are two vertical asymptotes $y = y_0, y_1$ between which $I(y)$ is negative. This means that for the infinitely long river we can obtain a non-uniform steady-progressing wave by letting y_1 correspond to the depth at $x = -\infty$ and y_0 to the depth at $x = +\infty$. If D and U are prescribed, y_0 and y_1 are uniquely determined and vice versa if y_0 and y_1 are prescribed then U and D are uniquely determined. The latter viewpoint is easier to adopt, that is, we find by setting $E = 0$ for $y = y_0$ and $y = y_1$:

$$(Uy_0 + D)^2 = S_y y_0^2 \left(\frac{y_0}{1 + \frac{2y_0}{B}} \right)^{4/3}$$

(II.14)

$$(Uy_1 + D)^2 = S_y y_1^2 \left(\frac{y_1}{1 + \frac{2y_1}{B}} \right)^{4/3} .$$

Equations (II.14) can easily be made linear to determine U and D simply. In the case we treated numerically $y_1 = 40$ ft, $y_0 = 20$ ft, we found that $U = 5$ mph.

Unfortunately, the steady-progressing waves do not exist in rivers, since the channels aren't uniform. Nevertheless, the speed of propagation, U , is very close to the observed speed of travel of flood waves. For this reason, some theoretical investigations are being made to establish the relationship between propagation speeds of transient phenomena and U . These results are needed to specify the accuracy of long range forecasts of flood conditions.